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IMPLICIT AND EXPLICIT ELASTOPLASTIC BEM FORMULATIONS APPLIED TO LOCALIZATION.

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Abstract. *In this paper an implicit BEM formulation is proposed to analyse elastoplastic two-dimensional bodies. The proposed algorithms are formulated as usual by assuming an initial stress or strain field applied over the domain area where plastic strains take place. The initial stress integrals are approximated over triangular and quadrangular cells. Integral representations are taken to represent the stress and strain field at internal points as well as along the boundary. An appropriate scheme to achieve the algorithmic tangent matrix is used together with the proposed implicit scheme. Numerical analyses are performed to show the accuracy of the results, also emphasising the differences between the analysed schemes. Classical elastoplastic examples are analysed to show the mesh dependency of the elastoplastic local model and the concentration of the plastic strain along narrow bands given by the cell sizes defined by the discretization. The numerical examples confirm the necessity of adopting other conditions to guarantee the results independent of the chosen mesh.*

Key words: *Boundary Elements, Non-linear Problems, Localization.*

1. INTRODUCTION

Along the last three decades, the Boundary Element Method (BEM) has proved to be appropriate to deal with an enormous number of engineering problems. The technique is nowadays a well-established procedure for the analysis of many practical engineering applications. In particular, the use of BEM to analyse non-linear problems has deserved special attention of the BEM community. By several reasons, fracture mechanics problems are by far the non-linear problem best treated by using BEM. In this case, only the crack lines have to be discretized, saving computer time and also increasing the accuracy of the results and the reliability of the global solution (Cruse, 1988) and (Aliabadi & Rooke, 1991).

Non-linear phenomena, such as plasticity, visco-plasticity and no-tension for instance, were treated by BEM in the earliy eighties, (Telles, 1983), (Venturini, 1983) and (Brebbia et al., 1984), after the correct obtaintion of the free term for the initial strain tensor made by Bui (1978). As the BEM formulations work on the stress space, it is expected that numerical solutions for non-linear analysis are better than other techniques that require differentiation of shape functions to compute the stress field. The boundary element has already proved to be able to compute high gradients and stress and strain concentrations. Thus, the BEM formulations might be recommended for non-linear analysis that exhibits the mentioned characteristics.

Although proving to give good results, the BEM non-linear approaches, appearing before this decade, were all based on the very simple explicit scheme accomplished by constant matrix procedures. The results obtained by using those simple models seemed to be precise and may suggest that BEM does not require to follow better approaches.

Implicit approaches have been proposed more recently. Jim et al.(1989) have used implicit integration for BEM finite deformation plasticity. Telles & Carrer (1991,1994) have also proposed an implicit model to solve elasto-plastic problems in the context of dynamic analysis for which they followed mass matrix approach. The CTO (Consistent Tangent Operator) has been recently introduced in the boundary element technique by Bonnet & Mukherjee (1996), using a scheme similar to the one proposed by Simo, & Taylor (1985) for finite elements.

One aspect forgotten by the boundary element community up to now is concerned with the strain localisation phenomena. This problem is certainly appropriate to be analysed by BEM; it exhibits small areas of interest inside the body, where the dissipation of energy occurs, as well as rather large displacement gradients. In the context of finite element analysis, strain localisation has been an important research subject for the improvement of numerical analysis of structure failures. However, there has been none or only limited interest to apply boundary element methods in the context (Maier et al., 1995).

Material behaviours characterised by constitutive relations that exhibit a softening branch (or a non-associated behaviour) bring great difficulties to classical (local) continuum theories in the description of localisation phenomena (Rudnicki & Rice, 1975), (Rice, 1976), (Benallal, 1988) and (Pijaudier-Cabot, & Benallal, 1993). The associated boundary value-problem is actually no longer mathematically well posed after the onset of localisation, and local continua allow for an infinitely small bandwidth in shear or in front of a crack tip (Pijaudier-Cabot, & Benallal, 1993) and (Benallal, & Tvergaard, 1995). At the numerical level, these difficulties translate into pathological mesh dependence of solutions (Bazant et al., 1984) and (Borst, 1988).

In this paper, explicit and implicit BEM formulations will be discussed regarding their ability of representing high plastic strains. A CTO is proposed to solve J_2 elasto-plastic problems. Examples are solved to illustrate that the BEM formulations, conveniently written, exhibit the pathological mesh dependence and enforce the dissipation zone to be reduced to an infinitely small bandwidth.

2 BASIC EQUATIONS FOR PLASTICITY

The following relations are used to define the flow theory of plasticity with isotropic hardening.

- The Cauchy stress tensor increment is given by:

$$\dot{\sigma} = E : (\dot{\varepsilon} - \dot{\varepsilon}^{p}) \tag{1}$$

where $\dot{\varepsilon}$ is the total strain rate, $\dot{\varepsilon}^p$ stands for the plastic strain rate and E is the matrix of elastic moduli;

- The yield criterion

$$f(\sigma, R(p)) \leq 0 \tag{2}$$

where R is the size of the yield surface and p the cumulated plastic strain defined by:

$$\dot{p} = \sqrt{\tfrac{2}{3}\dot{\varepsilon}^p : \dot{\varepsilon}^p} \tag{3}$$

- Plastic flow is given by the normality rule to the plastic potential F, i.e.

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial F}{\partial \sigma} \tag{4}$$

where λ is the plastic multiplier;

- The hardening rule

$$\dot{p} = \dot{\lambda}\partial F / \partial R \tag{5}$$

The plastic multiplier in equations (4) and (5) satisfies the Kuhn-Tucker conditions:

$$\dot{\lambda} \geq 0, \qquad f \leq 0, \qquad \dot{\lambda} f = 0 \tag{6a,b,c}$$

When λ is positive, it is obtained by the consistency condition, i.e.

$$\dot{f} = 0 \tag{7}$$

Then, using relations (1)-(5), one obtains easily:

$$\dot{\lambda} = \left(\frac{\partial f}{\partial \sigma} : E : \dot{\varepsilon}\right) \Big/ \left(h + \frac{\partial f}{\partial \sigma} : E : \frac{\partial F}{\partial \sigma}\right) \tag{8}$$

where, the plastic modulus h is defined by,

$$h = -\frac{\partial f}{\partial R}\frac{\partial R}{\partial p}\frac{\partial F}{\partial R} \tag{9}$$

Those relations are the basic ones required writing any BEM algorithm for local plasticity. Thus, they are considered for all the schemes discussed here.

3 INTEGRAL REPRESENTATION OF DISPLACEMENTS AND STRESSES

Let us first consider an elastic body associated with many possible elastic states satisfying the Navier's equations, i.e.:

$$(-L_{ij}u_j) = -C_{ijkl}u_{k,lj} = -Gu_{i,ll} - \frac{G}{1-2\nu}u_{l,li} = b_i \tag{14}$$

where u_k represents the components of the displacement field, G is the shear modulus and ν is Poisson's ratio.

For a domain Ω with boundary Γ, standard integral representations are derived by applying Betti's principle (Green's second identity). Particularly, displacement and stress integral representations are easily derived and may be found in Brebbia (1984):

$$c_{ik}u_k = -\int_\Gamma p^*_{ik}u_k d\Gamma + \int_\Gamma u^*_{ik}p_k d\Gamma + \int_\Omega u^*_{ik}b_k d\Omega \tag{15}$$

$$\beta\sigma_{ij} = -\int_{\Gamma} S_{ijk} u_k d\Gamma + \int_{\Gamma} D_{ijk} p_k d\Gamma + \int_{\Omega} D_{ijk} b_k d\Omega \tag{16}$$

p_k and b_k are traction and body force components respectively; the symbol "*" is related to the fundamental solution corresponding to a Dirac delta-type load applied in the collocation point (the second considered elastic state); the free terms c_{ik} and β are dependent upon the boundary geometry; D_{ijk} and S_{ijk} are kernels derived from equation (15).

For non-linear problems, Betti's principle can not be directly applied. Moreover, in plasticity, the state variables are history dependent. In this case, after splitting the total strain into its elastic and plastic components, the Navier operator applies only to the elastic part, therefore equation (14) becomes:

$$-L_{ij}\dot{u}_j = \dot{b}_i - C_{ijkl}\dot{\varepsilon}^p_{kl,j} \tag{17}$$

where ε^p_{kl} is the plastic strain tensor.

In equation (17), the plastic strains act as fictitious body forces. By interpreting the last term as an initial stress field, equation (17) becomes:

$$-L_{ij}\dot{u}_j = \dot{b}_i - \dot{\sigma}^p_{ij,j} \tag{18}$$

Thus, the Somigliana's identity (15), for plasticity based on the initial stress approach, can be expressed by:

$$c_{ik}\dot{u}_k = \int_{\Gamma} u^*_{ik}\dot{p}_k d\Gamma - \int_{\Gamma} p^*_{ik}\dot{u}_k d\Gamma + \int_{\Omega} u^*_{ik}\dot{b}_k d\Omega + \int_{\Omega} \varepsilon^*_{ijk}\dot{\sigma}^p_{jk} d\Omega \tag{19}$$

As for elasticity, the integral representation of stress rates can be obtained by differentiating (19) with respect to space co-ordinates and applying Hooke's law. Thus, one obtains,

$$\dot{\sigma}_{ij} = \int_{\Gamma} D_{ijk}\dot{p}_k d\Gamma - \int_{\Gamma} S_{ijk}\dot{u}_k d\Gamma + \int_{\Omega} D_{ijk}\dot{b} d\Omega + \int_{\Omega} E_{ijmk}\dot{\sigma}^p_{mk} d\Omega + g_{ij}\left(\dot{\sigma}^p_{mk}\right) \tag{20}$$

where the kernel E_{ijmk} comes from the differentiation of the plastic integral and $g_{ij}\left(\dot{\sigma}^p_{mk}\right)$ is a free-term that appears due to the strong singularity of this kernel.

Equation (20) was derived only for internal points. For boundary nodes one must find the limit when q, internal collocation point, goes to Q, on the boundary. Several schemes have already been discussed to derive the stress representation for boundary points. The simplest scheme very often adopted consists of writing only the algebraic representation using traction components (Cauchy's formula) and numerical differences of displacements. In order to obtain more accurate boundary stresses, an appropriate algebraic relation have been derived from the corresponding integral representation.

Although several alternatives could be followed, depending on the way the hypersingular terms are treated (reduced kernels, addition of an extra boundary value, etc., see Cruse, & Richardson, 1996), it has been decided to work on the hypersingular term to transform it into a regularised one. For this scheme, the continuity of the displacement derivatives at collocation points taken along the boundary must be assumed (Guiggiani, 1994). Using this alternative, points defined inside the elements have been defined to compute stresses along the boundary. It is important to mention that to complete the integral representation a new free term corresponding to boundary points has been derived as well. The free term for smooth boundary nodes is given by:

$$g_{ij}\left(\dot{\sigma}^{p}_{mk}\right) = -\frac{1}{8}\left(2(1+\nu)\dot{\sigma}^{p}_{ij} + (1-3\nu)\dot{\sigma}^{p}_{ll}\delta_{ij}\right) \qquad (21)$$

4 NUMERICAL FORMULATION

As it is well known, equations (19) and (20) of the precedent section can be transformed into algebraic representations by approximating u_k and p_k along the boundary duly divided into elements, as well as b_k and σ^{p}_{mk} over the domain now divided into cells. One can write as many algebraic equations as needed. Similarly, one can write an appropriate number of algebraic stress equations, the ones where the stress values are required to solve the problem. Moreover, without lost of generality, this description can be continued without using the rate symbol; It is important to note that the final algebraic representations to be achieved can be applied to corresponding rate or incremental problems.

Thus, using shape functions to approximate all variables, equations (19) and (20) become (Brebbia, 1984):

$$\mathbf{HU} = \mathbf{GP} + \mathbf{TB} + \mathbf{E}\boldsymbol{\sigma}^{\mathbf{p}} \qquad (22)$$

$$\boldsymbol{\sigma} = -\mathbf{H'U} + \mathbf{G'P} + \mathbf{T'B} + \mathbf{E'}\boldsymbol{\sigma}^{\mathbf{p}} \qquad (23)$$

where $\mathbf{U}$ and $\mathbf{P}$ are vectors containing the nodal values for displacements and tractions, respectively; $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{\mathbf{p}}$ are the stress and the initial stress vectors; $\mathbf{H}$, $\mathbf{H'}$, $\mathbf{G}$, $\mathbf{G'}$, $\mathbf{T}$, $\mathbf{T'}$, $\mathbf{E}$ and $\mathbf{E'}$ are the influence matrices arising from the numerical integration over elements and cells.

Applying the boundary conditions, equations (22) and (23) become

$$\mathbf{AX} = \mathbf{F} + \mathbf{E}\boldsymbol{\sigma}^{\mathbf{p}} \qquad (24)$$

$$\boldsymbol{\sigma} = -\mathbf{A'X} + \mathbf{F'} + \mathbf{T'B} + \mathbf{E'}\boldsymbol{\sigma}^{\mathbf{p}} \qquad (25)$$

$\mathbf{X}$ is the vector of boundary unknowns; $\mathbf{A}$ and $\mathbf{A'}$ contain the coefficients due to the unknown boundary values and $\mathbf{F}$ and $\mathbf{F'}$ are independent vectors due to prescribed boundary conditions and body forces.

Equations (24) and (25) can be reduced to:

$$\mathbf{X} = \mathbf{M} + \mathbf{R}\boldsymbol{\sigma}^{\mathbf{p}} \qquad \boldsymbol{\sigma} = \mathbf{N} + \mathbf{S}\boldsymbol{\sigma}^{\mathbf{p}} \qquad (26a,b)$$

where **M** and **N** are the elastic solutions (displacements and stresses); **R** and **S** represent the influences of the initial stresses.

Equations (26) can be adopted for the solution of any non-linear problem and conveniently modified to solve anisotropic problems; see for instance (Chaves, Fernandes & Venturini, 1999), where the similar equations are adopted to solve varying thickness plate in bending.

For elastoplastic solutions, one must realise that the plastic stress increments are computed following the proper elastoplastic relations given in section 2.

Telles & Carrer (1991, 1994) were ones of the first to propose an implicit model to solve elasto-plastic problems. They proposed an algorithm based on a continuous tangent operator. After that Bonnet & Mukherjee have used, by the first time for BEM, the concept of consistent tangent operator together with an initial strain approach.

The formulation implemented in this work is similar the one proposed by Bonnet, but conveniently modified to work with initial stress fields.

Initially, a scheme obtain the implicit return to the plastic surface was implemented. The Von Mises criterion has been adopted for the constant hardening case. For this situation, the procedure is easily derived. The prevision is defined interactively, starting by assuming the load increment entirely elastic, which is than corrected (correction step) to guarantee the radial return to the plastic surface. This return algorithm is adopted to achieve the local tangent operator $\mathbf{C}^{\mathbf{epC}}$ to be used to correct the global consistent matrix:

$$C^{epC} = \partial\ \sigma^{arr} / \partial\Delta\ \varepsilon \tag{27}$$

where σ^{arr} comes from the return algorithm.

Equation (26b) can be written in its incremental form. Then, the plastic stress tensor $\boldsymbol{\sigma}^{\mathbf{p}}$ is now added to its both sides to give:

$$\{\mathbf{Y}(\Delta\boldsymbol{\varepsilon}_{\mathbf{n}})\} = -[\mathbf{C}]\{\Delta\boldsymbol{\varepsilon}_{\mathbf{n}}\} + \{\mathbf{N}_{\mathbf{n}}\} + [\overline{\mathbf{S}}]\{[\mathbf{C}][\Delta\boldsymbol{\varepsilon}_{\mathbf{n}}] - [\Delta\boldsymbol{\sigma}_{\mathbf{n}}]\} = \mathbf{0} \tag{28}$$

where [**C**] is the usual Hookean elastic tensor, $[\overline{\mathbf{S}}] = [\mathbf{S}] + [\mathbf{I}]$ with [**I**] being the identity matrix, Δ indicates increments and the subscript n gives the increment number.

The stress and strain tensor increments, $\Delta\boldsymbol{\varepsilon}_{\mathbf{n}}$ and $\Delta\boldsymbol{\sigma}_{\mathbf{n}}$, that cumulate into the stress and strain values at interaction n, lead to their up-dated values, as follows.

$$\boldsymbol{\varepsilon}_{\mathbf{n+1}} = \boldsymbol{\varepsilon}_{\mathbf{n}} + \Delta\boldsymbol{\varepsilon}_{\mathbf{n}} \quad \text{and} \quad \boldsymbol{\sigma}^{\mathbf{arr}}_{\mathbf{n+1}} = \boldsymbol{\sigma}_{\mathbf{n}} + \Delta\boldsymbol{\sigma}_{\mathbf{n}} \tag{29a,b}$$

where $\boldsymbol{\sigma}^{\mathbf{arr}}_{\mathbf{n+1}}(\Delta\boldsymbol{\varepsilon}_{\mathbf{n+1}})$ is achieved by the return algorithm and $\boldsymbol{\sigma}_{\mathbf{n}}$ is given by the BEM algebraic relation.

Rearranging equation (28) to take into account relations (29), the following expression is found:

$$\{\mathbf{Y}(\Delta\boldsymbol{\varepsilon}_{\mathbf{n}})\} = -[\mathbf{C}]\{\Delta\boldsymbol{\varepsilon}_{\mathbf{n}}\} + \{\mathbf{N}_{\mathbf{n}}\} + [\overline{\mathbf{S}}]\{[\mathbf{C}][\Delta\boldsymbol{\varepsilon}_{\mathbf{n}}] - \{\boldsymbol{\sigma}^{\mathbf{arr}}_{\mathbf{n}}\} + [\Delta\boldsymbol{\sigma}_{\mathbf{n}}]\} = \mathbf{0} \tag{30}$$

Equation (30) can be solved using the Newton Raphson scheme. For that one needs to define:

$$\Delta\boldsymbol{\varepsilon}^{\mathbf{i+1}}_{\mathbf{n}} = \Delta\boldsymbol{\varepsilon}^{\mathbf{i}}_{\mathbf{n}} + \delta\Delta\boldsymbol{\varepsilon}^{\mathbf{i}}_{\mathbf{n}} \tag{31}$$

where the superscript i gives the iteration.

Considering only the first variation of $\{\mathbf{Y}(\Delta\boldsymbol{\varepsilon}_{\mathbf{n}})\}$, equation (30) becomes:

$$\{\mathbf{Y}(\Delta\boldsymbol{\varepsilon}_{\mathbf{n}})\} = \left[[\mathbf{C}] - [\overline{\mathbf{S}}]\{[\mathbf{C}] - [\partial\boldsymbol{\sigma}^{\mathbf{arr}\ \mathbf{i}}_{\mathbf{n}} / \partial\Delta\boldsymbol{\varepsilon}^{\mathbf{i}}_{\mathbf{n}}]\}\right]\{\delta\Delta\boldsymbol{\varepsilon}^{\mathbf{i}}_{\mathbf{n}}\} \tag{32}$$

where the term between parenthesis in the right hand side is the consistent tangent operator for BEM elastoplastic formulation, i.e.,

$$[\mathbf{C}^{\mathbf{epC}}] = \left[[\mathbf{C}] - [\overline{\mathbf{S}}]\{[\mathbf{C}] - [\partial\boldsymbol{\sigma}^{\mathbf{arr}\ \mathbf{i}}_{\mathbf{n}} / \partial\Delta\boldsymbol{\varepsilon}^{\mathbf{i}}_{\mathbf{n}}]\}\right] \tag{33}$$

Note that, for the initial stress algorithm, the elastic constitutive law must multiply the results of equation (32).

5. NUMERICAL ANALYSIS

After discussing the ways to treat elastoplastic solids by using BEM, it is important to illustrate them by analyzing some classical examples. The main objective of this work is to show that BEM procedures are accurate enough to analyze elastoplastic solids, and to show that for softening material the localization phenomena can be precisely captured. Constant and tangent matrix procedures are employed to solve the example to prove that the second one is far better to capture plastic strain concentrations. The mesh dependence is clearly detected by using the CTO proposed by BEM, as well.

Stretching rectangle. For this analysis, a simple rectangle is considered to show the localization phenomenon and the mesh dependence problem. Figure 1 gives the adopted sizes, the basic boundary and internal discretizations and an internal small region where reduced yielding stresses are assumed. Other finer discretizations have been used to build the displacement x reaction curves and to illustrate the mesh dependence. The finest discretization, exhibiting 2048 cells has been used to obtain all the results presented here. In order to observe the localization phenomenon, a weaker zone near the center has been assumed. The yielding stresses are equal to 2.0Mpa everywhere except over the weaker internal square. The yielding stress is reduced of 0.2Mpa at the central node of that small zone. For other inside points, the reduction varies linearly from the center to the square region boundary. The Young modulus assumed was E= 2,000.00N/mm^2, while assuming h=-0.01234E defines the softening behaviour. The rectangle is loaded by applying the boundary displacement field shown in Figure 1; no displacement is enforced along the left support, while along the right end δ=-0.24mm was assumed. Several increment sizes have also been tested to analyze how the developed procedure behaves.

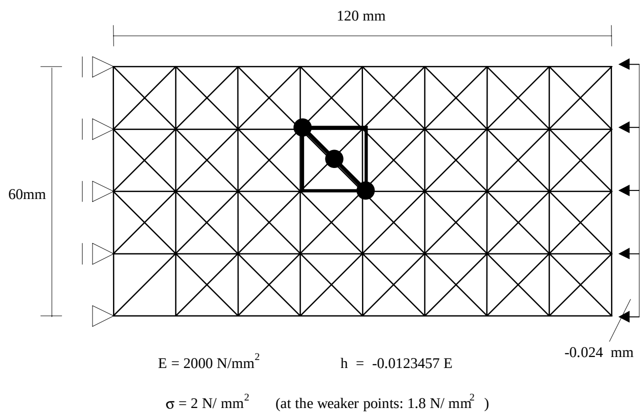

Figure 1. Rectangle: Size and discretizations.

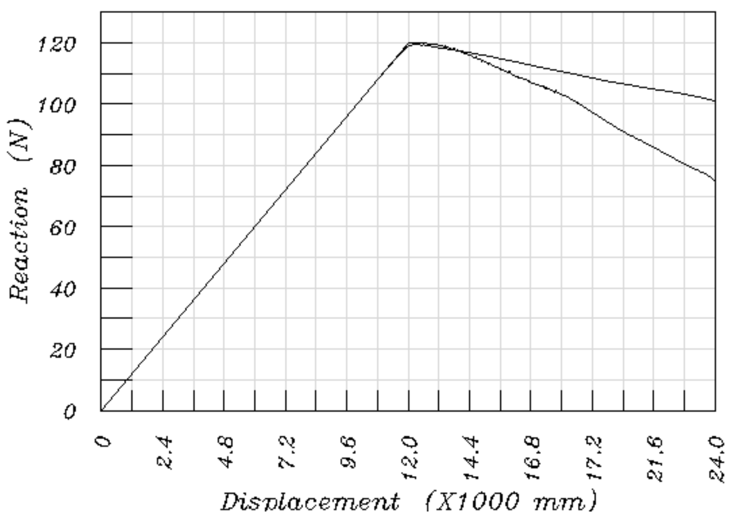

Figure 2. Reaction x displacement curve for the 512 and 2048 cell cases.

Figure 2 gives the displacement x reaction curves captured for the two finest discretizations, 512 and 2048 cells, respectively. The total reactions along the clamped end were computed by integrating the obtained tractions. The mesh dependence is clearly demonstrated in this figure. Using other courser meshes different curves, for the softening branch, have been captured. The results obtained by following the consistent tangent operator if far better when compared with the ones computed by the classical constant matrix BEM. Using the standard scheme the mesh dependence can also be detected, but the results are more unstable in comparison with the previous scheme.

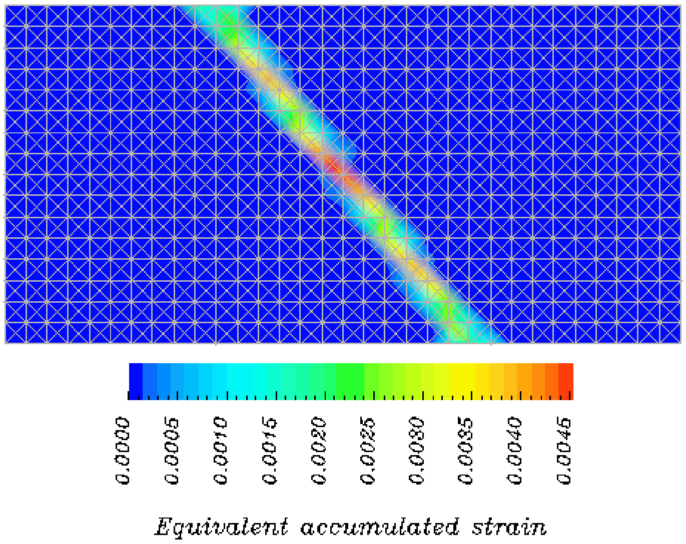

Figure 3. The final equivalent plastic strain over the body for the finest mesh.

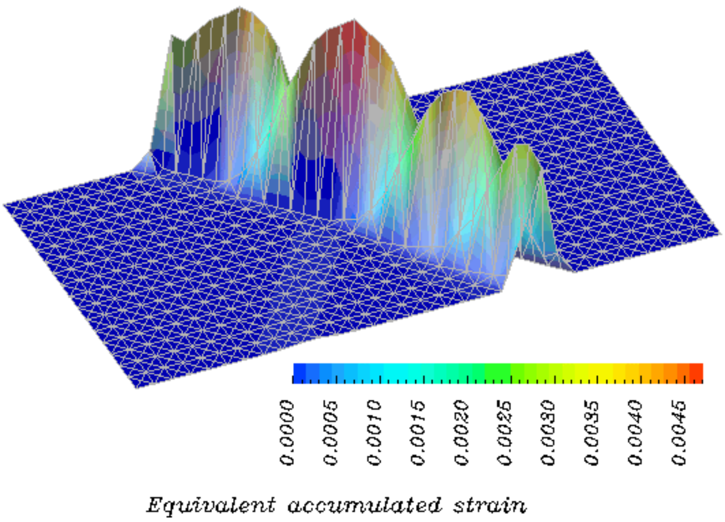

Figure 4. 3D visualization for the final equivalent plastic strain field.

Another important aspect of this BEM analysis is concerned with the final plastic strain configuration achieved. Again, the solution obtained by using the CTO scheme is the expected one, with a clear localized diagonal narrow zone. This narrow zone is always precisely defined over a row of cells. Thus, the width of this zone is exactly the mesh size. This solution is given in Figure 3, where the concentration of the plastic strain over the diagonal narrow band across the body is clearly defined. No plastic strain develops out of that zone. On the other hand, using the classical BEM approach with constant matrix the localized zone can be achieved, but they are not well defined (for some discretization more than one localized zones have been detected). For all discretizations tested, plastic strains developed over a much wider zone were obtained, when the explicit scheme was used together with constant matrix.

In order to visualize better the quality of the solution obtained by using the CTO scheme, a 3D representation of the total effective plastic strain is given in Figure 4. These results illustrate again the localized plastic zone and how large can be the values of plastic strains. The formulation developed has proved to be able to deal with this kind of problem even when characterized by stress or strain concentrations. This kind of solution has been captured for all discretizations experimented when the CTO scheme was adopted.

Figure 5 illustrates as well the capability of the proposed BEM scheme. One can see that stress field is disturbed only over the narrow dissipation zone. In addition, this figure shows that the body is split into two elastic parts with no displacement due to plastic strain evolution inside the narrow band.

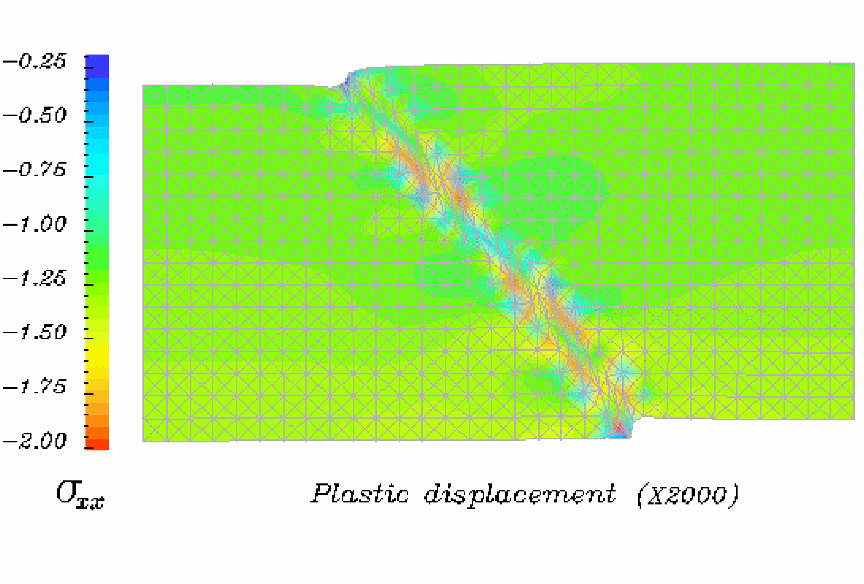

Figure 5. Plastic displacements and normal stress component σ_{xx} distribution.

6. CONCLUSIONS

Implicit and explicit elasto-plastic BEM formulations have been discussed. The implicit case scheme was proposed together with the definition of a consistent tangent operator. Both formulations have been tested to verify their capability of achieving accurate stress field for problems characterized by exhibiting strain concentrations and employing course and fine discretizations. It has been found that only implicit BEM based on CTO scheme is able to capture precisely plastic strain concentrations. This scheme was also able to exhibit clearly the classical problem of mesh dependence. On the other hand the standard procedure, based on explicit scheme and constant matrix does not present the same accuracy, therefore it is not recommended to investigate elasto-plastic problems exhibiting plastic strain concentrations.

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